Corporate default behavior: A simple stochastic model

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We compare observed temporal dynamics of corporate default to a first-passage-time model and find that corporations default as if via diffusive dynamics.

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I. INTRODUCTION

An entity defaults when it fails to make a contractually obligated payment to a creditor. When that entity is a corporation the impact on securities issued by that corporation can be both swift and dramatic. Consequently, the dynamics of the default process bear directly on the practice of risk management and security pricing as well as on current research on asset price dynamics. While there is a long tradition of asset dynamics research in both the finance and physics literature [1–4], the closely related dynamics of default have receive comparatively little attention. In this paper we show that a simple stochastic model of corporate default dynamics that is well known in statistical mechanics—geometric Brownian motion with an absorbing barrier—provides a remarkably good description of observed default behavior of corporations and implies that corporations default as if via diffusive dynamics.

While default forecasting has been the subject of active research for decades [5–7], the relationship between observed corporate default data and the stochastic dynamics of firm value remains indirectly explored. This is due, in part, to the historical development of the two major research programs in this area. One program can be characterized as originating with the work of Altman [8,9] where a credit score is developed as a linear function of explanatory accounting variables. While this approach has been successful in predicting default and is consistent with what one might expect given the focus of rating agencies on financial ratios [10,11], it has a linear deterministic structure that provides limited insight into the stochastic dynamics of corporations. The other major research program began with the work of Merton [12], who applied a contingent-claims approach [13,14] to the calculation of corporate bond spreads. Although this approach is an explicitly stochastic treatment of corporate dynamics, it suffered marginalization because it failed to reproduce observed corporate bond prices. Current bond price research [15] now recognizes that this marginalization was not justified because the difference in price dynamics between corporate and government bonds is only partially determined by corporate default dynamics. Thus, while extensions of Merton's model (e.g., [16–18]) that typically involve multiple stochastic processes can reproduce bond prices, they provide limited direct information regarding the stochastic dynamics of firm value. Nevertheless, the success of these models and of commercial products based on proprietary stochastic models of default[1] provides compelling evidence in support of the usefulness of a stochastic approach to explaining default dynamics.

Our understanding of corporate default dynamics is complicated by the comparatively rare nature of the corporate default event. Fortunately, a variety of financial institutions including rating agencies have compiled cumulative default probabilities for corporations with credit ratings. Ratings provide the investor with a measure of the default risk of corporate securities. Indeed, it has been known for decades that "a rating is designed to indicate how likely it is that the issuer will be able to meet principal and interest payments" [10]. In many respects ratings are rather remarkable in that the complexity of the capital structure of a firm can be summed up in a single letter grade.[2] That this scalar representation of the capital structure is, as cited above, a measure of the probability of payment (and, by necessity, the probability of default) implies that a comparatively simple dynamics may be able to describe default dynamics, and the purpose of this paper is to demonstrate that this is indeed so.

The challenge posed to a theoretical description of corporate default is illustrated in Fig. 1 where we present the cumulative default probability as a function of time for AAA, BBB, and CCC rated companies published by Standard and Poor's [23]. The AAA data denoted by the diamonds are roughly convex for all time less than 10 years. Beyond 10 years there are no observed defaults and the cumulative default probability is constant. The CCC data denoted by the triangles are quite different with a concave function for all time. The BBB data show characteristics of both AAA and CCC: convex for short times and concave for long times. Furthermore, we see that in passing from AAA to CCC the cumulative default probabilities change by an order of mag-

[1]See, for example, KMV's CREDITMONITOR™ [19], J. P. Morgan's CREDITMETRICS™ [20], and Moody's public firm risk model [21].

[2]In this paper we have used the data published by Standard and Poor's and, consequently, their rating convention. In this convention a AAA rating is the rating of "highest grade" and a D rating is reserved for a firm in default and indicative of the lowest estimated salvage value. AAA, AA, A, and BBB are all considered "investment grade." Ratings BB, B, CCC, and CC are known by the names "not investment grade," "high yield," and "junk." The rating C is reserved for income bonds. Finally, ratings DDD–D are in "default, with rating indicating relative salvage value" [22].

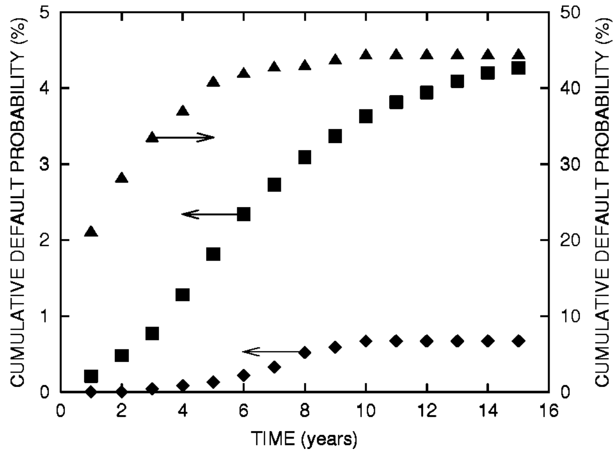

FIG. 1. Observed cumulative default probability for AAA, BBB, and CCC credits published by Standard and Poor's (diamonds, squares, and triangles, respectively). The arrows point to the appropriate *y* axis for each data series.

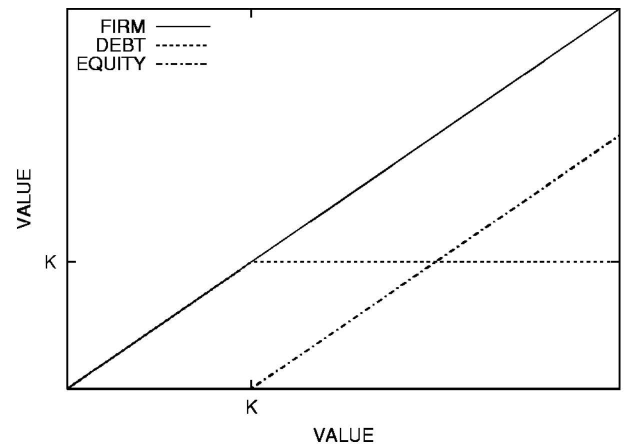

FIG. 2. A schematic representation of a simple corporate capital structure.

nitude. To describe this default behavior we develop an analytic expression for default probability in Sec. II and apply it to the observed default data published by Standard and Poor's [23] in Sec. III. Our analysis will demonstrate that a single variable in the analytic formula provides effective discrimination between various credit ratings. We conclude this paper in Sec. IV.

II. THE DEFAULT MODEL

The structural basis for the firm model is, perhaps, best illustrated by the levered mutual fund or unit trust described by Crosbie [24]: a firm totally invested in traded securities. We assume that the assets of the firm are securities with readily observable market prices with which the value of the firm can be measured at any time. To buy these assets the firm can access two sources of funds. First, the firm can obtain funds from the equity market by taking in cash from the shareholders in the firm. Second, the firm can obtain funds from the debt market by issuing a bond or taking out a loan of amount K. The process of generating debt is always accompanied by the generation of a credit rating. This can be explicit and public as seen when a firm requests a rating of a debt issue by a rating agency, explicit and private as when a bank makes a loan, or implicit and public when a nonrated issue is priced in the bond market. We assume that the firm invests this cash (shareholder equity and debt) in the assets described above and that in, say, one year's time the assets are sold, the loan is repaid, and any remaining funds are distributed among the shareholders. This payout is illustrated in Fig. 2 where the value of the firm V (solid line) is decomposed into the equity and debt components (dot-dashed and dashed/solid lines, respectively). If the value of the assets V is less than[3] K then the equity holders get nothing and the firm is in default as it can only pay $(V/K)\%$ of the debt amount. If the value of the assets V is equal to K the equity holders also get nothing, but the debt can be fully repaid. Finally, if the value of the assets V exceeds K then the debt can be repaid and the value of the equity is equal to the difference $V-K$. Thus the value of the firm V is equal to the value of the equity, $\max(V-K,0)$, and the value of the debt, $V-\max(V-K,0)$. Given these functional forms for the payout to the stockholders and debt holders of the firm, it is a relatively straightforward exercise in option-pricing theory[4] to price the value of either debt or equity at any point in time between the initial investment and the payout date. This idealized "wait and see where the assets end up" model and the reality of debt agreements can be brought closer together by allowing for the debt holders to reorganize the firm whenever V reaches a prescribed level:[5] a common feature of contractual agreements between the debt holders and the firm known as bond indentures. Although corporations typically have a more complex capital structure and often invest in assets for which prices are hard to come by, the basic features of equity, debt, and (by implication) bankruptcy are, in general, the same: funds are raised in the equity and debt markets and if a debt payment is missed default occurs.

The model begins with the traditional assumption that a corporation is represented as an asset with a market value V, and that, while the return of the asset is uncertain because of various risks associated with the business, it is lognormally distributed, namely,

$$\frac{dV}{V}=\mu dt+\sigma dW, \tag{1}$$

[3]Strictly speaking, the firm would also need to pay the interest on the debt as well, i.e., $K(1+r)$ where r is the annual rate of interest of the debt.

[4]In the language of option pricing, equity is a "call option" on the value of the firm with a "strike price" at the level of the debt K. Similarly, debt is a "buy-write" where the debt holders have effectively bought the firm V and simultaneously sold a call to the equity holders.

[5]See, for example, [25]. The variable K can be a function of the actual level of the debt and can be time dependent.

where μ and σ are the constant drift and volatility of the asset value, t denotes time, and W is a standard Brownian motion.[6] This geometric Brownian motion is the ''harmonic oscillator'' of the contingent-claims theory of asset dynamics and, consequently, a reasonable starting point. However, a considerable literature has developed in recent years demonstrating the need to go beyond this description of dynamics for traded securities (particularly at very short times) and we shall return to this below in Sec. IV. Following Black and Cox [25] we also assume that when the asset value falls to a prescribed level denoted by K the company defaults. Transforming to the normalized variable q, defined by

$$q \equiv \frac{1}{\sigma} \ln\left(\frac{V}{K}\right), \tag{2}$$

and using Ito's lemma, we have that

$$dq = \mu^* dt + dW, \tag{3}$$

where $\mu^* \equiv (\mu/\sigma - \sigma/2)$. The default level now becomes $q = 0$. Since q is a measure of how far the firm is from the default level, it has a natural interpretation as the distance to default discussed by Crosbie [24,26]. Given the initial value q_0, we can calculate the expected cumulative default probability $D(t)$ from the first-passage-time probability [27,28]

$$D(t) = N\left(\frac{-q_0 - \mu^* t}{\sqrt{t}}\right) + e^{-2\mu^* q_0} N\left(\frac{-q_0 + \mu^* t}{\sqrt{t}}\right), \tag{4}$$

where $N(x)$ is the cumulative normal distribution function.[7] The model for ratings-based default embodied by this expression can be interpreted as illustrated in Fig. 3. When a firm is initially rated $(t=0)$ it will be q_0 standard deviations away from default. As time passes $(t>0)$ the company's credit state, buffeted by the vaguaries of the economic environment, diffuses with drift μ^* and unit volatility. Should the firm's fortunes evolve such that q becomes zero, it encounters the absorbing boundary of default. We now consider whether this model can describe observed default behavior.

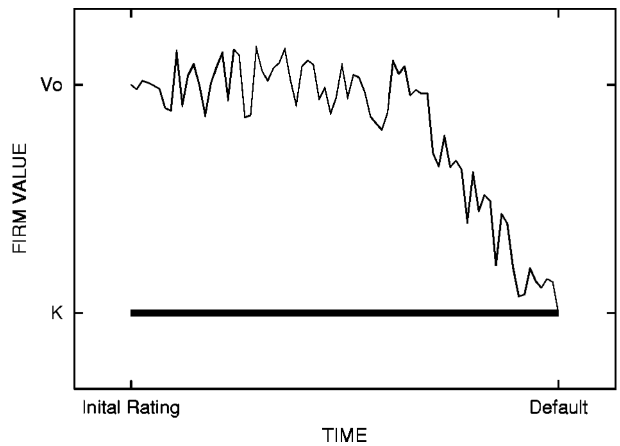

FIG. 3. A representation of the temporal evolution of a corporation from initial agency rating to default.

III. THE DEFAULT MODEL AND OBSERVED DEFAULT BEHAVIOR

There are two parameters in Eq. (4), q_0 and μ^*, that can be varied to fit the observed default probabilities for each rating. As an example of how this expression can be used to represent observed default behavior we consider the published static pool average cumulative default probabilities for each credit rating given by Standard and Poor's [23] shown in Table I. These values represent the probability of default as a function of time following the *initial* rating of the company. For example, while a company that was initially rated BBB may undergo any number of rating changes over time, once it defaults it is treated in this analysis as being a BBB default. It can be seen that for each rating the change in the cumulative default probability slows (and in some cases ceases) around 10 years, reflecting the fact that few defaults (and in some cases no defaults) have been observed beyond 10 years. This, as discussed below, is most likely due to the limitation of the sample sizes available. Consequently, we used data from the first eight years for each rating to fit Eq. (4). The parameters q_0 and μ^* were obtained by minimizing the sum of the squared difference between the observed default behavior shown in Table I and the calculated values obtained from Eq. (4) subject to the constraint that the long-time cumulative default probability $D(\infty) = \exp[-2\mu^* q_0]$ be

TABLE I. Observed cumulative default probabilities (%) as reported by Standard and Poor's.

Time (yr)	AAA	AA	A	BBB	BB	B	CCC
1	0.00	0.01	0.04	0.21	0.91	5.16	20.93
2	0.00	0.04	0.11	0.48	2.82	10.90	28.04
3	0.04	0.11	0.18	0.77	5.00	15.36	33.35
4	0.08	0.21	0.31	1.28	7.04	18.60	36.83
5	0.13	0.33	0.47	1.81	8.82	20.95	40.67
6	0.22	0.49	0.63	2.34	10.68	22.65	41.83
7	0.33	0.64	0.82	2.73	11.71	24.08	42.64
8	0.52	0.76	1.02	3.09	12.78	25.32	42.86
9	0.59	0.84	1.25	3.37	13.71	26.29	43.63
10	0.67	0.90	1.48	3.63	14.42	27.13	44.23
11	0.67	0.94	1.68	3.81	15.19	27.54	44.23
12	0.67	0.98	1.78	3.94	15.55	27.76	44.23
13	0.67	0.98	1.84	4.09	15.84	27.83	44.23
14	0.67	0.98	1.88	4.20	15.84	27.83	44.23
15	0.67	0.98	1.92	4.27	15.84	27.83	44.23

[6]The notion of the value of the firm as a time-dependent stochastic variable can be traced at least as far back as the pioneering options work of Black and Scholes [13] and Merton [14], and is a basic tenet of essentially all contingent-claims security analysis.

[7]Those familiar with the work of Black and Cox [25] will see a strong similarity between our Eq. (4) and their Eq. (7). There is, indeed, a direct correspondence that can be derived by taking their reorganization boundary to be independent of time and noting that their Eq. (7) is for the probability that the firm has not defaulted while our Eq. (4) is for the probability that the firm has defaulted.

TABLE II. Calculated cumulative default rates (%) obtained by fitting Eq. (4) to the data in Table I.

Time (yr)	AAA	AA	A	BBB	BB	B	CCC
1	0.00	0.00	0.00	0.01	0.35	4.24	19.19
2	0.00	0.01	0.03	0.25	2.54	11.11	29.49
3	0.02	0.07	0.14	0.77	5.08	15.72	34.60
4	0.07	0.19	0.31	1.35	7.27	18.84	37.66
5	0.15	0.34	0.50	1.88	9.06	21.06	39.69
6	0.25	0.49	0.67	2.34	10.51	22.71	41.13
7	0.36	0.64	0.83	2.73	11.70	23.97	42.20
8	0.47	0.78	0.97	3.05	12.68	24.97	43.02
9	0.58	0.90	1.09	3.31	13.49	25.76	43.66
10	0.69	1.01	1.19	3.53	14.17	26.41	44.17
11	0.79	1.10	1.27	3.72	14.76	26.94	44.59
12	0.88	1.19	1.35	3.87	15.25	27.38	44.93
13	0.96	1.26	1.41	4.00	15.68	27.76	45.22
14	1.04	1.32	1.46	4.11	16.05	28.07	45.46
15	1.11	1.37	1.50	4.20	16.37	28.35	45.66

ordered as expected [i.e., $D_{AAA}(\infty) \leqslant D_{AA}(\infty) \leqslant \cdots \leqslant D_{CCC}(\infty)$]. This multidimensional minimization was effected using the generalized reduced gradient (GRG2) nonlinear optimization solver in Microsoft EXCEL$^{\mathrm{TM}}$. The fitted default probabilities using Eq. (4) and the parameters q_0 and μ^* resulting from the fitting procedure just described are shown in Table II.

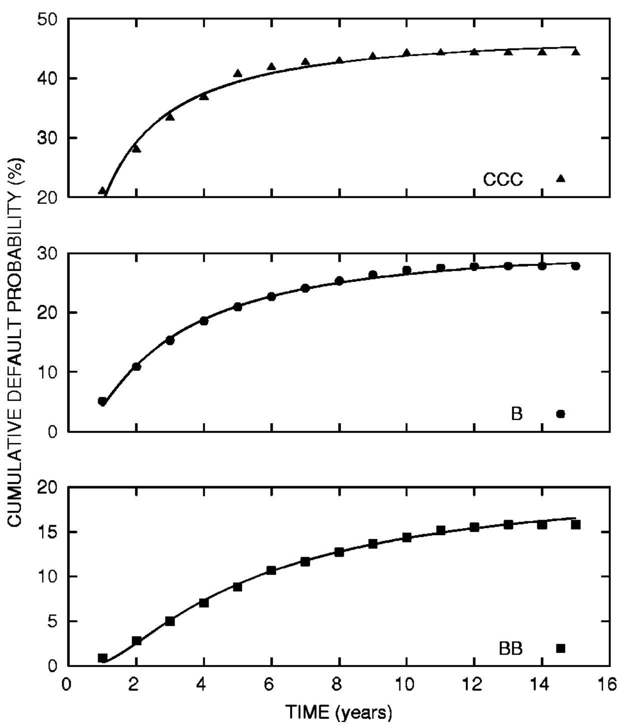

FIG. 4. Observed and calculated cumulative default probability for noninvestment grade credits. The observed probabilities denoted by the symbols are from the Standard and Poor's report. The model curves are based on a fit of Eq. (4) to the observed probabilities between one and eight years.

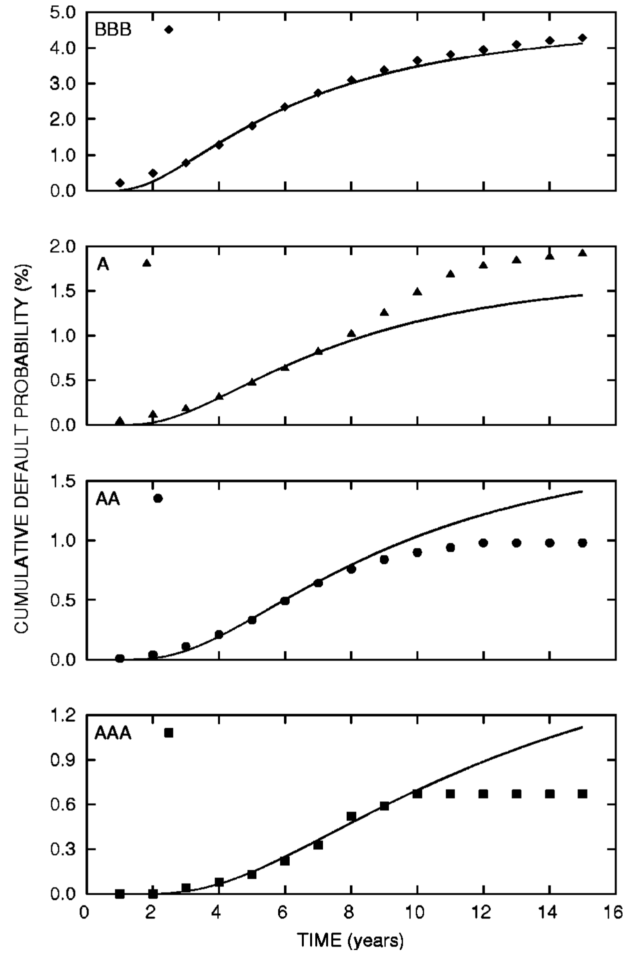

FIG. 5. Observed and calculated cumulative default probability for investment grade credits. The observed probabilities denoted by the symbols are from the Standard and Poor's report. The model curves are based on a fit of Eq. (4) to the observed probabilities between one and eight years.

The result of this fit for non-investment grade credits is shown in Table II and Fig. 4. The fit to the these credits illustrates the rather good ability of this two-parameter model to describe the default behavior over the entire 15 year horizon based on a fit to only the first eight years of data. The concave nature of the data is well reproduced by Eq. (4) as is the substantial slowing of default probability accumulation beyond 10 years. A comparison of Tables I and II shows, however, that beyond a certain time horizon there are no observed defaults and the cumulative default probability ceases to change. The model, however, continues to rise gradually, indicating that the lack of observed defaults is likely due to a limited sample size and that, in time, we can expect to see more defaults in this area. Financially, there is nothing that would account otherwise for the observed lack of defaults.

The result of our fit for investment grade credits is shown in Table II and Fig. 5. The BBB credits shown in this figure illustrate an interesting qualitative change in default behavior: the short-horizon data are *convex* in time. The intermediate-horizon data are linear in time and the longer-horizon data are concave. We see that our simple two-

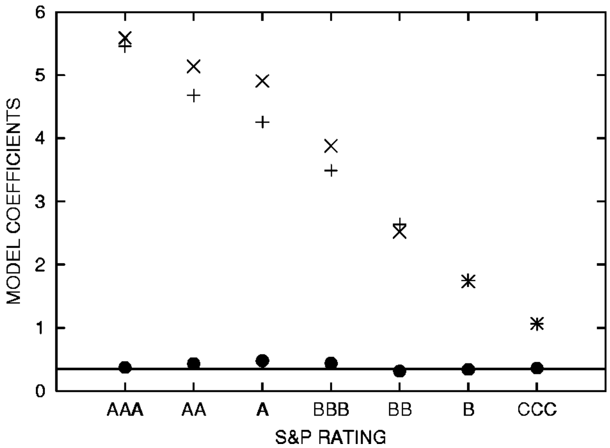

FIG. 6. The calculated initial distance to default q_0 ($\times$ and $+$) and drift μ^* (● and line) as a function of credit rating. The (q_0, μ^*) pairs $(+, ●)$ were obtained by an independent fit for each rating of Eq. (4) to the $t \leqslant 8$ yr data in Table I. The q_0 results $\times$ were obtained by a simultaneous fit of Eq. (4) to all of the $t \leqslant 8$ yr data in Table I, which allowed for a different value of q_0 for each rating but a single value of μ^* for all ratings. The resulting value of μ^*, 0.35 is depicted by the horizontal line.

TABLE III. Calculated and observed mean time to default (yr).

Rating	Variable μ^*	Constant μ^*	Observed
AAA	14.7	16.1	8.0
AA	10.8	14.8	8.3
A	9.0	14.1	8.2
BBB	8.0	11.2	6.6
BB	8.4	7.2	4.7
B	5.1	5.0	3.4
CCC	3.0	3.1	3.2

parameter model provides a very reasonable description of the data. The cumulative default behavior of A credits is a challenge to the model. The model clearly tracks the observed data well between one and eight years and from 11 to 15 years, but an accumulation of defaults in the 8 to 11 year time frame results in a substantial offset between the observed and the calculated data in the 11 to 15 year region. One solution is to fit over the first 10 years. While this is perfectly reasonable and most people using such a model to fit their data would likely do so, we felt it useful to see how far we could get using a uniform time range for fitting purposes. For AA and AAA credit default behavior we see good fits to observations over the first eight years and reasonable extrapolations with significant deviations coinciding with the point at which the cumulative default data stop changing due to lack of observed defaults.

The coefficients of Eq (4), q_0 and μ^*, that resulted from the fitting procedure that generated Figs. 4 and 5 are shown as a function of credit rating in Fig. 6 together with those coefficients that resulted from a subsequent simulation suggested by the initial results. The symbols $+$ and ● correspond to the fitted values of q_0 and μ^* as described in the previous paragraphs. We see the intuitively expected result that the better the credit rating the larger the initial distance to default. Recalling from our earlier discussion that the distance to default is measured in standard deviations, we see that the average AAA company is initially about 5.5 standard deviations from default, the average BBB company is initially about 4 standard deviations from default, and the average CCC company is initially about 1 standard deviation from default.

The normalized drift μ^*, denoted by ●, resulting from the fitting procedure that generated Figs. 4 and 5 is remarkable in that, while each credit rating was fitted individually, these normalized drifts are quite similar. This similarity prompted us to explore the results that would follow if μ^* were assumed *a priori* to be the same for all credit ratings. Under this constraint differences in default behavior among various ratings are driven solely by the initial distance to default q_0. The results of a global fit with this restriction are shown in Fig. 6 with q_0 now given by the symbol $\times$ and $\mu^* = 0.35$ shown as a horizontal line in Fig. 6. Comparing the q_0 for constant and variable drift we see that setting μ^* constant across credit ratings has essentially no impact on q_0 for the lower credit ratings and has a minor impact on the higher credit ratings. Indeed, with the exception of the AA, A, and BBB credits, the results for q_0 are nearly identical as indicated by the coincidence of the symbols $+$ and $\times$. That the default dynamics of all credit ratings can be represented by a single value of μ^* implies that differences in cumulative default behavior among various ratings are indeed driven almost exclusively by the initial distance to default q_0.

Modeling the default process as a first-passage time yields a simple expression for the mean time to default: q_0/μ^*. We compare the results of this expression with those reported by Standard and Poor's in Table III. The deviation between the calculated and observed results reflects the lack of observed default at longer tenors. However, for the same reason that we would expect the longer-tenor cumulative default probability for the AA and AAA credits to increase over time, so too do we expect the mean time to default to increase over time for investment-grade credits.

IV. SUMMARY

Comparing observed corporate cumulative default probabilities to those calculated using a stochastic model, we find that corporations default as if via diffusive dynamics. The model, based on a contingent-claims analysis of corporate capital structure [25], yields a single analytic expression for corporate default behavior that is calibrated easily with historical default probabilities. We used this model to analyze the observed default data published by Standard and Poor's [23] and found that a single variable in the analytic formula provides effective discrimination between various credit ratings. This variable is quite similar to the ''distance to default'' described by Crosbie [24,26] and provides an attractive interpretation of the default process in terms of the bond indenture analysis of Black and Cox [25]. Despite its simple underpinnings, the model is remarkably successful in describing the cumulative default rates published by Standard and Poor's [23]. This implies that the capital structure of corporations, despite their differences, map onto the simple "effective" capital structure given in the Merton model. The ability to represent observed default behavior by a single analytic expression and to differentiate credit-rating-dependent default behavior with a single variable recommends this model for a variety of risk management applications including the mapping of bank default experience to public credit ratings.

Geometric Brownian motion was proposed as a description of firm dynamics roughly 30 years ago [12–14,25] and before cumulative default statistics had been collected. While it does rather well in describing observed cumulative defaults, it is undoubtedly an incomplete description of firm dynamics; particularly so at short time horizons. As the depiction of equity in Fig. 2 illustrates, equity is a function of the value of the firm. Thus, the price fluctuations of individual companies are directly related to the dynamics of firm value, and recent research (e.g., [29]) has clearly demonstrated that a dynamics richer than geometric Brownian motion underlies these fluctuations. An integration of the dynamics implicated in price fluctuation research with cumulative default results should provide a much more realistic description of the default process.

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[1] J.-P. Bouchaud and M. Potters, *Theory of Financial Risks: From Statistical Physics to Risk Management* (Cambridge University Press, New York, 2000).
[2] *The Random Character of Stock Market Prices*, edited by P. H. Cootner (MIT Press, Cambridge, MA, 1964).
[3] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, New York, 2000).
[4] J. Voit, *The Statistical Mechanics of Financial Markets* (Springer-Verlag, New York, 2001).
[5] *Managing Credit Risk: The Next Great Financial Challenge*, edited by J. B. Caouette, E. I. Altman, and P. Narayanan (John Wiley & Sons, New York, 1999).
[6] A. Saunders, *Credit Risk Measurement: New Approaches to Value at Risk and Other Paradigms* (John Wiley & Sons, New York, 1999).
[7] E. I. Altman, *Corporate Distress and Bankruptcy: A Complete Guide to Predicting and Avoiding Distress and Profitting from Bankruptcy* (John Wiley & Sons, New York, 1993).
[8] E.I. Altman, J. Financ. **23**, 589 (1968).
[9] E.I. Altman, R. Haldeman, and P. Narayanan, J. Banking Financ. **1**, 29 (1977).
[10] H. C. Sherwood, *How Corporate and Municipal Debt is Rated: An Inside Look at Standard & Poor's Rating System* (John Wiley & Sons, New York, 1976).
[11] Standard & Poor's 2000 Corporate Ratings Criteria (2000), www.standardandpoors.com.
[12] R.C. Merton, J. Financ. **29**, 449 (1974).
[13] F. Black and M. Scholes, J. Political Econom. **81**, 637 (1972).
[14] R. Merton, Bell J. Econom. Manage. Sci. **4**, 141 (1973).
[15] J. Bohn, A Survey of Contingent-Claims Approaches to Risky Debt Valuation', www.kmv.com.
[16] I. Kim, K. Ramaswamy, and S. Sundaresan, Financial Management **22**, 117 (1993).
[17] D. Shimko, N. Tejima, and D. van Deventer, J. Fixed Income **3**, 58 (1993).
[18] F.A. Longstaff and E.S. Schwartz, J. Financ. **50**, 789 (1995).
[19] CREDITMONITOR™, www.kmv.com.
[20] G. M. Gupton, C. G. Finger, and M. Bhatia, *CREDITMETRICS™—Technical Document* (J. P. Morgan & Co., New York, 1997).
[21] J. R. Sobehart and R. M. Stein, *Moody's Public Firm Risk Model: A Hybrid Approach to Modeling Short Term Default Risk* (Moody's Investors Service, Global Credit Research, New York, 2000).
[22] J. C. V. Horne, *Financial Market Rates and Flows*, 4th ed. (Prentice Hall, Englewood Cliffs, NJ, 1994).
[23] L. Brand and R. Bahar, *Special Report: Ratings Performance 1999* (Standard and Poor's, New York, 2000), p. 10.
[24] P. Crosbie, *Credit Derivatives: Trading and Management of Credit and Default Risk* (John Wiley & Sons, Singapore, 1998), pp. 299–315.
[25] F. Black and J.C. Cox, J. Financ. **31**, 351 (1976).
[26] P. J. Crosbie, Modeling default risk, www.kmv.com 999-0000-031.
[27] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 2nd ed., Springer Series in Synergetics Vol. 13 (Springer-Verlag, New York, 1985).
[28] J. E. Ingersoll, *Theory of Financial Decision Making*, Rowman & Littlefield Studies in Financial Economics (Rowman & Littlefield, Savage, MD, 1987).
[29] V. Plerou, P. Gopikrishnan, L. Amaral, M. Meyer, and H. Stanley, Phys. Rev. E **60**, 6519 (1999).